Complete Model of a Self-gravitating Cosmic String I. A New Class of Exact Solutions and Gravitational Lensing

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Abstract

We find solutions of Einstein's field equation for topologically stable strings associated with the breaking of a U(1) symmetry. Strings form in many GUTs and are expected whenever the homotopy group $\Pi_1(M_0)$ is non-trivial. The behavior of the fields making up the string is described by the Euler-Lagrange equations. These fields appear in the energy-momentum tensor so we must solve simultaneously for the coupled Einstein-scalar-gauge field equations. Numerical results are obtained using a Taylor-series method. We obtain a 5-parameter family of solutions and discuss their physical characteristics. Significant gravitational lensing can occur due to strings based on this model and are shown for different solutions. Finally, we prove that the assumption of regularity at the string axis is not necessary by looking at the physical properties of the string solutions.

1 Introduction

Phase transitions occur in the early universe as a consequence of its expansion and cooling. The transitions cannot be observed directly but can be inferred from the theory of groups and symmetries in elementary particle physics. If these phase transitions do occur, there will necessarily form topological defects during the transition. Theses defects take their name after their characteristic of being trapped regions of "old symmetry" surrounded by "new symmetry". In essence, they retain the characteristics of the state of the universe as it was before the phase transition. The topology of the defect varies according to the symmetry group G characterizing the fields present in the universe before the symmetry breaking and the symmetry group H which describes the symmetry of the field after the symmetry breaking, i.e. H includes all elements of G which leave the vacuum expectation value of the scalar field invariant. When the first homotopy group is nontrivial, $\Pi_1(G/H) \neq I$, i.e. the topological knot cannot be "unwound", the topology of the defect is linear, or string like. Topologically stable strings occur in non-Abelian gauge theories such as SU(5) (Shafi and Vilenkin [1]) and SO(10) (Kibble *et al.* [2]) grand unified theories but also in the simple case of Abelian U(1) symmetry.

The gravitational field of strings has been studied in general relativity starting with the work of Vilenkin [3], subsequent to the Newtonian approach to topological defects given by Zel'dovich *et al.* [4] and Kibble [5]. He used the linear approximation of general relativity and an energy-momentum tensor which has no lateral stresses but only terms describing the energy density and the pressure (tension) along the axis of the string. This assumption was used later in work such as that of Gott [6] and Hiscock [7] in looking for exact solutions of Einstein's equations for a string where, in addition, the energy density of the string was taken to be constant. It has since been shown by Raychaudhuri [8] that the Gott and Hiscock solution is not consistent with proper boundary conditions.

Subsequently, there have been a number of attempts to obtain better models of cosmic strings. These range from treatments which impose a fixed background geometry wherein the properties of the string are calculated, to the attempt of Laguna-Castillo and Matzner [9] and Garfinkle and Laguna [10] who alternately held the metric functions fixed while integrating the string equations and held the string field properties fixed while computing the new metric from the Einstein equations.

The purpose of this paper is to extend previous attempts at describing gravitating cosmic strings by simultaneously solving the coupled Einstein-scalar-gauge field equations, so that our model will include the effect of the energy-momentum of the string on the background. In addition, we want to study the physical properties of the string solutions that emerge from this set of differential equations. Recently, Shaver [11] has

examined the equations for non-stationary cosmic strings and solved them for the simple energy-momentum tensor introduced by Vilenkin. Based on the general set of equations obtained by Shaver and dropping the Vilenkin restrictions to the form of the energy-momentum tensor, we will use a numerical integration technique to attempt to find all possible solutions for self-gravitating strings.

2 The coupled Einstein-scalar-gauge field equations

We will be studying string topological defects associated with the spontaneous symmetry breaking of an Abelian group G=U(1). The Lagrangian of this Abelian-Higgs model is:

$$L = -\frac{1}{2}(D^{\mu}\Phi)^{*}(D_{\mu}\Phi) - V(\Phi) - \frac{1}{2}F_{\mu\nu}F^{\mu\nu} \tag{1}$$

where $F_{\mu\nu} = \nabla_{\mu}A_{\nu} - \nabla_{\nu}A_{\mu}$, $D_{\mu} = \nabla_{\mu} + ieA_{\mu}$, $V(\Phi)$ is the potential of the scalar field, A_{μ} is the gauge field, e the gauge coupling constant, and Φ the scalar field. The symmetry breaking potential has the form $V(\Phi) = \lambda(\Phi^{*}\Phi - \eta^{2})^{2}$ where λ is the self coupling constant of the Higgs field and η is the value of the symmetry breaking Higgs field. Shaver [11] has shown that for the specific choice of the energy-momentum tensor given by Vilenkin [3], $\lambda = e^{2}/8$. Since we are concerned with finding solutions where T_{r}^{r} and T_{ϕ}^{ϕ} are not necessarily zero, we will set $\lambda = \alpha e^{2}/8$ where α is a constant and can be taken as a free parameter. Note that, when acting on a scalar field, ∇_{μ} simply becomes ∂_{μ}. Also, due to the symmetries of the Christoffel symbols, $\nabla_{\mu}A_{\nu} - \nabla_{\nu}A_{\mu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}$.

The spontaneous symmetry breaking is obtained by introducing the Higgs field. The symmetry of the system after such a breaking is then determined by the degeneracy of the vacuum expectation value of the scalar field:

$$\begin{aligned} \Phi &= |\Phi|e^{i\theta}\ , \quad 0 \leq \theta \leq 2\pi && (2) \\ &= Re^{i\theta} && (3) \end{aligned}$$

where $R \equiv |\Phi|$.

To describe the gauge field, we refer to the work of Nielsen and Olesen [12] who obtain from their second equation of motion for flat space-time:

$$\begin{aligned} A_{\mu} &= \frac{1}{e^{2}}\frac{j_{\mu}}{|\Phi|^{2}} - \frac{1}{e}\partial_{\mu}\theta && (4) \\ &= \frac{1}{e}[P-1]\phi_{,\mu} && (5) \end{aligned}$$

where $j_\mu \equiv \partial^\nu F_{\mu\nu}$, $P \equiv j_\mu / e|\Phi|^2$, and we have set $\theta = \phi$, the angular coordinate (see below). (We can now rewrite the equation for the scalar field as $\Phi = Re^{i\phi}$.) These expressions for A_μ and Φ will be used in our equations below and we will use R and P to characterize the scalar and gauge field respectively.

The equations of motion are given by:

$$\frac{\partial L}{\partial \chi} = \nabla_\mu \frac{\partial L}{\partial(\partial_\mu \chi)} \tag{6}$$

where χ is replaced by R, θ, or A_μ to give the three Euler-Lagrange equations. When A_μ is replace by equation (5), the equation involving the derivatives with respect to θ is immediately satisfied, which is a consequence of our earlier choice of $\theta = \phi$. The covariant derivative ∇_μ requires the metric of the space-time to be specified.

We start by assuming that the space-time defined by the self-gravitating string has general cylindrical symmetry. The metric has the form:

$$ds^2 = -e^{2(K-U)}(dt^2 - dr^2) + e^{-2U} W^2 d\phi^2 + e^{2U} dz^2 \tag{7}$$

where U, K and W are unknown functions of r only and will be solved for in our calculations.

Using this metric, the Euler-Lagrange equations become:

$$e^{2(U-K)} \left(R'' + R' \frac{W'}{W} \right) - RP^2 \frac{e^{2U}}{W^2} - \frac{dV}{dR} = 0 \tag{8}$$

$$e^{2(U-K)} \left(P'' - P' \left(\frac{W'}{W} - 2U' \right) \right) - e^2 R^2 P = 0 \tag{9}$$

where $' \equiv \partial / \partial r$.

The energy-momentum tensor for the Lagrangian (1) is given by:

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}} \left\{ \frac{\partial(\sqrt{-g}L)}{\partial g^{\mu\nu}} - (\frac{\partial(\sqrt{-g}L)}{\partial g^{\mu\nu}_{,\gamma}})_{,\gamma} \right\} \tag{10}$$

The components of T^μ_ν are the following:

$$T^t_t = T^z_z = -\frac{e^{2(U-K)}}{2} \left[R'^2 + \frac{e^{2K}}{W^2} R^2 P^2 + \frac{e^{2U}}{W^2 e^2} P'^2 + 2V e^{2K-2U} \right] \tag{11}$$

$$T^r_r = \frac{e^{2(U-K)}}{2} \left[R'^2 - \frac{e^{2K}}{W^2} R^2 P^2 + \frac{e^{2U}}{W^2 e^2} P'^2 - 2V e^{2K-2U} \right] \tag{12}$$

$$T^\phi_\phi = \frac{e^{2(U-K)}}{2} \left[-R'^2 + \frac{e^{2K}}{W^2} R^2 P^2 + \frac{e^{2U}}{W^2 e^2} P'^2 - 2V e^{2K-2U} \right] \tag{13}$$

The Einstein equations, $G^\mu_\nu = 8\pi T^\mu_\nu$, that couple the string stress-energy to its space-time geometry become:

$$-2\frac{W''}{W} + 2K'\frac{W'}{W} - 2U'^2 = 8\pi\left[R'^2 + \frac{e^{2K}}{W^2}R^2P^2 + \frac{e^{2U}}{W^2e^2}P'^2 + 2Ve^{2K-2U}\right] \quad (14)$$

$$2K'\frac{W'}{W} - 2U'^2 = 8\pi\left[R'^2 - \frac{e^{2K}}{W^2}R^2P^2 + \frac{e^{2U}}{W^2e^2}P'^2 - 2Ve^{2K-2U}\right] \quad (15)$$

$$2K'' + 2U'^2 = 8\pi\left[-R'^2 + \frac{e^{2K}}{W^2}R^2P^2 + \frac{e^{2U}}{W^2e^2}P'^2 - 2Ve^{2K-2U}\right] \quad (16)$$

$$-2\frac{W''}{W} + 4U'\frac{W'}{W} + 4U'' - 2U'^2 - 2K'' = 8\pi\left[R'^2 + \frac{e^{2K}}{W^2}R^2P^2 + \frac{e^{2U}}{W^2e^2}P'^2 + 2Ve^{2K-2U}\right] \quad (17)$$

These equations can be rearranged to give a simpler set of equations. The first simplification comes from equations (14) and (17), from which we obtain $K' = 2U' + \frac{C_1}{W}$. Since $W(0) = 0$ (see initial conditions), we set $C_1 = 0$ otherwise $K'(0) \to \infty$ and we run into numerical problems (assuming U' remains finite at the origin). Therefore, we directly obtain a relationship between K and U which is of the form $K = 2U + C_2$, where a new constant of integration is introduced. Since all solutions may be transformed such that $K(0) = U(0) = 0$, we set $C_2 = 0$ and therefore $K = 2U$.

By subtracting equation (14) from (15) we obtain an expression for W'':

$$W'' = -8\pi\left(\frac{e^{2K}R^2P^2}{W} + 2Ve^{2(K-U)}W\right) \quad (18)$$

Using $K' = 2U'$, and (15) and (16) gives for U'':

$$U'' = 4\pi\left(\frac{P'^2e^{2U}}{e^2W^2} - 2Ve^{2(K-U)}\right) - U'\frac{W'}{W} \quad (19)$$

The field equations are integrable only if the conservation equations of energy and momentum $T^{\mu\nu}_{;\nu} = 0$ are satisfied. These conservation laws often give, analogous to the first integrals of classical mechanics, an important indication of how to solve the field equations. Using the energy-momentum tensor given above, the only non-vanishing component of the conservation equations is:

$$R'\left(e^{2(U-K)}\left(R'' + R'\frac{W'}{W}\right) - RP^2\frac{e^{2U}}{W^2} - \frac{dV}{dR}\right) +$$
$$P'\frac{e^{2U}}{e^2W^2}\left(e^{2(U-K)}\left(P'' - P'\left(\frac{W'}{W} - 2U'\right)\right) - e^2R^2P\right) = 0 \quad (20)$$

Note that this equation is a linear combination of equation (8) and (9) and therefore indicates that one of equation (8) and (9) can be taken as redundant. Since the conservation

of energy equation was derived from the Einstein equations, one of these equations can also be taken as redundant. We will take equation (15) as the redundant equation, which we will nevertheless continue using as a consistency check for our numerical integration, and keep both equations (8) and (9). Equation (15) can be rewritten as:

$$R'^2 = \frac{1}{4\pi}\left(K'\frac{W'}{W} - U'^2\right) + \frac{e^{2K}}{W^2}R^2P^2 - \frac{P'^2e^{2U}}{e^2W^2} + 2Ve^{2(K-U)} \tag{21}$$

A complete set of equations consists of equations (8), (9), (18), and (19), with equation (21) above. Similar results have been obtained by Garfinkle [13] and Laguna-Castillo and Matzner [9] (hereafter Garfinkle and LCM respectively).

Putting $V = \alpha\frac{e^2}{8}(R^2-\eta^2)^2$, $K = 2U$, $e^U = X$, and rescaling R by η, W and r by $\sqrt{8}/\eta e$ gives us the final set of equations to solve:

$$\begin{aligned}
X'' &= \frac{1}{2}\pi\eta^2X^3\left(\frac{P'^2}{W^2} - 16\alpha(R^2-1)^2\right) + X'\left(\frac{X'}{X} - \frac{W'}{W}\right) && (22)\\
W'' &= -8\pi\eta^2X^2\left(X^2\frac{R^2P^2}{W} + 2\alpha W(R^2-1)^2\right) && (23)\\
R'' &= X^4\frac{RP^2}{W^2} + 4\alpha X^2R(R^2-1) - R'\frac{W'}{W} && (24)\\
P'' &= 8X^2R^2P + P'\left(\frac{W'}{W} - 2\frac{X'}{X}\right) && (25)\\
R'^2 &= \frac{1}{4\pi\eta^2}\frac{X'}{X}\left(2\frac{W'}{W} - \frac{X'}{X}\right) + X^2\left(X^2\frac{R^2P^2}{W^2} - \frac{P'^2}{8W^2} + 2\alpha(R^2-1)^2\right) && (26)
\end{aligned}$$

Notice that the "e"s all disappear from the equations. It can be interpreted as a scaling factor. We are left with a 2-parameter set of equations instead of 3. The units that we chose to adopt in this paper are the natural units: $c = \hbar = G = 1$. In the original equations, R had units $[M]$, W and r, $[L]$, and P and X, $[1]$. After rescaling, the units of R,W and r have become $[1]$ like the other variables. The dimensions of η, and e, λ and α are $[M]$ and $[1]$ respectively. According to these units, the energy density will be expressed in terms of $[M]^4$.

3 Integration Method and Solutions

For any given set of initial conditions, the differential equations determine the behavior of R, P, X and W as a function of r. Although all solutions obtained from any initial set

of conditions are valid solutions of the differential equations, we will examine in this work only solutions that exhibit the particular asymptotic behaviour, for which $\lim_{r\to\infty} R = 1$ and $\lim_{r\to\infty} P = 0$. These conditions are derived from the requirement of finite action. Such solutions will be denoted as "acceptable" solutions. They have the structure of a trapped vortex or string. An example of an "acceptable" solution and an "unacceptable" solution are portrayed in figure 1.

A Taylor-series method was used to numerically integrate the set of five coupled differential equations. This method was chosen in part because of the flexibility of implementation at various orders, which allowed the same computer code to be used for both rapid exploration of parameter space followed by a more detailed study at a higher order of integration. In addition the use of a Taylor scheme, with the production of the higher order Taylor coefficients, allows direct calculation of many of the geometrical objects of interest (such as the Christoffel symbols or curvature tensors), without any need for data fitting or divided difference differentiations. Because of the complexity of the functions to be integrated, we used the REDUCE algebraic computing system to produce the mathematical expressions for the high order derivatives needed for the numerical integration. The transformation of the algebraic expressions to build the Taylor integration program in the C source code was done using the SCOPE package in REDUCE. The code so generated was verified to be correct by running the C source code back through REDUCE. This latter check is particularly important when using automated code production systems. Several consistency checks were applied to our solutions to ensure accuracy of the results. For example, the value of R' computed by the integration scheme was compared with that obtained from the redundant expression shown in equation (26).

The solutions of the differential equations are uniquely determined by a set of initial conditions. Recall that the variable U has been defined such that $U(0) = 0$ and therefore $X(0) = 1$. We also require that $R(0) = 0$, i.e. the axis is the region of false vacuum where the potential attains its local maximum. In order to define an axis, we must have $W(0) = 0$. Previous studies have required regularity on the axis and have imposed the condition that $\lim_{r\to 0} g_{\phi\phi}/r^2 = 1$ (Garfinkle [13]). This means, in terms of our choice of coordinates, that $W'(0) = 1$. However, it is not clear that this condition is justified. For now, we will set $W'(0) = 1$ and will return to the case $W'(0) \neq 1$ in the next section.

By asking that all derivatives be finite at the origin, we require that $P(0) = W'(0)$ and $X''(0) = \frac{\pi\eta^2}{4}\left(\frac{P''(0)^2}{W'(0)^2} - 16\alpha\right)$ while $W'(0)$, $R'(0)$, $P''(0)$ are undertermined and $X'(0)$, $W(0)$, $W''(0)$, $R''(0)$, $P'(0)$ vanish. An "acceptable" solution is therefore derived from those initial conditions, given the five free parameters η, α, $W'(0)$, $R'(0)$, and $P''(0)$.

The "acceptable" solutions form a particular subset of all solutions. The "goodness" of a solution is the deviation from the correct asymptotic structure and is measured by

computing $\sqrt{R'^2 + P'^2}$ at large r. In agreement with Shaver, an "acceptable" solution is found at $\alpha = 1.0$, $\eta = 0.19947106$, $W'(0) = 1.0$, $R'(0) = 1.4586085$ and $P''(0) = -4.0$ with a deviation of 0.00003625 measured at $r_{max} = 4.0$. From the precision on the parameters, one can see that to find an "acceptable" solution requires a very fine tuning of the initial parameters. A small deviation in one of the input parameters causes a large deviation at r_{max} and gives an "unacceptable" solution. Finding "acceptable" solutions is therefore very difficult. Because of the "sharpness" of the valley of deviation of an "acceptable" solution (see figure 2 and 3), or, in other words, because of the precision required for the initial parameters, it is impossible to search randomly all parameter space in a reasonable time and hope to fall on an "acceptable" solution. The approach we took for searching for "acceptable" solutions relies on the perturbation method of a known "acceptable" solution. By manually tuning the initial parameters for a perturbed solution and by using an extrapolation method to find the next "acceptable" solution, we were able to step away from the known solution and find the other "acceptable" solutions in the parameter space of all solutions. The subset obtained using this method covers the surfaces shown in figure 4 and 5. The ranges of the α and η parameters are such that α covers the region from 0.001 (α cannot be zero) to 2.5 and η, from 0.0 to 0.2.

For each new solution (from now on we will use "solution" to mean "acceptable" solution) we can compute the metric tensor, angular deficit α_D, energy-momentum tensor, Weyl tensor and Kretschmann scalar using the Taylor coefficients. We now examine the change in the physical properties of the string as one moves in parameter space. A representative sample of solutions are shown in figure 6. The angular deficit α_D (not to be confused with our input parameter α) is defined by $\Delta\phi = 2\pi(1 - \alpha_D)$, where α_D is the derivative of $\sqrt{g_{\phi\phi}}$ with respect to r. For our metric, $\alpha_D = (W'X - WX')/X^2$. If X and X' are set equal to 1, we recover Shaver's case. The first families of solutions, those with α constant, have $P''(r = 0)$ constant as well and show the same energy-momentum tensor. The special case where $\alpha = 1.0$ and $P''(0) = -4.0$ yield solutions which have no angular and radial energy-momentum components. This special case was studied in detail in Shaver. It was found that the α_D depends on η and that keeping α_D positive required that $\eta^2 < 4\pi$. The maximum value for η for these particular solutions is therefore ~ 0.2821 and relevant functions for this solution are displayed in figure 6. The angular deficits measured for this set of solutions are in agreement with Shaver and LCM. For all computed energy-momentum tensor components, our values agree with Shaver and Garfinkle but disagree by a factor of 10 from the values obtained by LCM. It was also noted that LCM obtain a value of 3%, which is also what we obtain, for their value of $1 - e^A$ (in our case, $1 - X^2$) at $\eta = 0.01$ and $\alpha = 1/4$ (corresponding to our value of $\alpha = 4$) but quote a value of 0.03% in the text. It should be noted that a factor of 100 in this expression does make this solution meaningfully different from Minkowski spacetime.

For the same families of solution, we also notice, in agreement with LCM, that the

angular deficit and Weyl tensor are small for values of $\eta < 0.01$ (see figure 6) and get larger and more important for $\eta > 0.01$ with increasing η. We also observe that the angular deficit and Weyl tensor increase with α. Changing the value of the symmetry breaking Higgs field η is like changing the shape of the potential. By increasing η, we are allowing the potential to be wider and the defect to contain more energy. This shows up in an increase in angular deficit and Weyl effect. The maximum η is reached when the angular deficit becomes 2π (or α_D becomes zero).

Examination of the solutions in figure 6, showing non-zero T_r^r and T_ϕ^ϕ components, confirms that the assumptions made in early works on strings are not always satisfied. Simply rescaling the potential of the scalar field by α changes the shape of the energy-momentum tensor and the spacetime quite dramatically. The energy-momentum tensor with zero angular and radial components gives a potential with $\alpha = 1$. Moving away from that particular state by scaling the potential up or down, introduces extra components of the energy-momentum tensor which we can attempt to explain in the following way. When the scalar field acquires more potential energy and this energy becomes dominant, the particles in the field will show a higher interaction which can be interpreted as tension in the string. The negative components of the energy-momentum tensor appearing in the corresponding solution correspond to a negative pressure, i.e. a tension in the r and ϕ directions. Decreasing the potential energy would mean, by the same argument, that the particles would be less bound together and this is shown as positive components of the energy-momentum tensor for the low potential solution. There seems to exist a transition region where the ϕ component of the energy-momentum tensor changes sign. This is probably related to the angular deficit growing larger as r increases and its action on the dynamics of the scalar field since the r component of the energy-momentum tensor does not show the same behavior.

4 Gravitational lensing from cosmic strings

The solutions we have derived introduce the possibility of the existence of cosmic strings in our universe which would curve spacetime around them and therefore provide some gravitational perturbation to their surroundings. The way to study this gravitational perturbation is to follow the null and timelike geodesics in the string spacetime and examine their behaviour and consequence for cosmological models.

The gravitational perturbation of cosmic strings on light rays produces real gravitational lensing and brings a physical (in contrast with geometrical) prediction for observational tests for cosmic strings. We first examine the null geodesics in the string spacetime

of each solution.

The string metric written in terms of the new coordinates is:

$$ds^2 = -X^2(r)(dt^2 - dr^2) + \frac{W^2(r)}{X^2(r)}d\phi^2 + X^2(r)dz^2 \tag{27}$$

and the null geodesic equation can be expressed as:

$$\frac{-E^2}{X^2(r)} + \frac{X^6(r)}{W^2(r)}L^2(\frac{dr}{d\phi})^2 + \frac{X^2(r)}{W^2(r)}L^2 = 0 \tag{28}$$

The geodesics are described in the plane $z = constant$. We can rewrite this equation in terms of functions of r_c, the closest approach radius where $dr/d\phi = 0$. This gives:

$$\frac{d\phi}{dr} = \pm\frac{1}{\frac{W(r_c)}{X(r_c)^2}h\sqrt{h^2-1}} \tag{29}$$

where $h = \frac{W}{W(r_c)}/(\frac{X}{X(r_c)})^2$. The $\pm$ sign in front of the expression on the right hand side of the equation can be interpreted as the receding part of the orbit (+ sign) and the approaching part of the orbit (− sign).

Therefore, for the receding part of an orbit starting at r_c and finishing at large $r = r_{max}$, the angle traversed can be computed using the integral:

$$\Delta\phi = \int_{r_c}^{r_{max}} \frac{1}{\frac{W(r_c)}{X(r_c)^2}h\sqrt{h^2-1}} \tag{30}$$

For the case where $W(r)/X(r) = \alpha_D r$, this integral can be solve analytically and, as expected, there is no angular deviation in the conical spacetime:

$$\Delta\phi = \frac{1}{\alpha_D}cos^{-1}(\frac{r_c}{r}) \;|_{r_c}^{r_{max}} \tag{31}$$

Therefore, when $\sqrt{g_{\phi\phi}}$ is proportional to r, the metric is conformally flat and one can apply a coordinate transformation and interpret the difference from Minskowski spacetime by the presence of a missing or surplus angle. In the case when the same metric coefficient is a a nonlinear function of r, there is no uniform conformal transformation which will make the metric flat. Thus the metric has real curvature and the string is a true gravitating string.

We use an 8-point Gauss-Legendre integration method to compute the integrated orbit of a photon in the equatorial plane of the string. Care is required in the integration near the closest approach, due to the vanishing denominator, but this can be handled easily

by using the limiting form near closest approach as given in equation (31). The orbits are plotted for a representative string solution and are shown for different closest approach radius in figure 7.

In discussing the null orbits, it is important investigate the possible existence of an event horizon or a photon cylinder (in analogy to the photon sphere for the Shwarzschild spacetime) associated with our string solutions. To determine if the solutions we have found have a photon cylinder, we need to find the zeroes of $dr/d\phi$. Examination of this function graphically shows that there is exactly one zero for each ray, that at closest approach, but no others. Thus there can be no rays which spiral continuously into the string axis, and thus there is no photon cylinder. We also find that there is no event horizon by considering the possibility of nullity of the appropriate Killing vector.

We are also interested in the Weyl tensor for different gravitating string solutions to determine whether or not there exist tidal effects near the string, since these can lead to an understanding of the distortion, shear, and rotation of geodesics near the string. All non-vanishing components of the Weyl tensor can be obtained from the following component:

$$C_{trtr} = \frac{2X''XW - 6X'^2W + 4X'W'X - W''X^2}{6W} \tag{32}$$

$$\tag{33}$$

using the relations: $C_{rzrz} = -C_{trtr}$, $C_{tztz} = -2C_{trtr}$, $C_{t\phi t\phi} = (\frac{W}{X^2})^2 C_{trtr}$, $C_{\phi z\phi z} = -C_{t\phi t\phi}$, and $C_{r\phi r\phi} = 2C_{t\phi t\phi}$. The fact that the Weyl tensor has non-zero components (see figure 6 for representative functions) combined with the non-vanishing of the Ricci tensor (as seen through the energy-momentum tensor) leads to the conclusion that these string solutions will produce significant gravitational lens effects. These effects are in strong contrast to the simple "prism" optical effect introduced by the traditional vacuum strings where both the Ricci and Weyl tensors are identically zero. Thus there will be real distortions and amplification of distant objects seen along lines of sight passing near these strings. Of course, in the static situation treated here, there can be no perturbations to the temperatures of distant sources, such as the cosmic background radiation.

5 Regularity at the Origin or is $W'(0) \neq 1$

In starting to look for solutions, we chose to set $W'(0) = 1$. This has been consistently assumed throughout the literature on cosmic strings, on the basis that $W'(0) = 1$ implies that the axis of the string is regular. On the other hand, there seem to be few arguments

that lead to a justification of this particular assumption of regularity. We will demonstrate in this section that this assumption of regularity at the axis of the cosmic string is not necessary and that in fact, the physics of the spacetime is the same if the angular deficit is put concentrated near the axis or if it is put at large r.

The first thing we want to do is ensure that there is no real singularity at the string axis. By singularity we mean the divergence of the Kretschmann scalar defined by $\mathcal{K} = \mathcal{R}^{\sigma\tau\mu\nu}\mathcal{R}_{\sigma\tau\mu\nu}$ where $R_{\sigma\tau\mu\nu}$ is the Riemann tensor. For the general metric given above, the Kretschmann scalar is given by:

$$\begin{aligned}
\mathcal{K} \;=\; & \frac{4}{X^8W^2}(3X''^2X^2W^2 - 10X''X'^2XW^2 + 6X''X'W'X^2W && (34)\\
& -2X''W''X^3W + 14X'^4W^2 - 22X'^3W'XW + 6X'^2W''X^2W && (35)\\
& +11X'^2W'^2X^2 - 6X'W''W'X^3 + W''^2X^4) && (36)
\end{aligned}$$

We observe that the Kretschmann scalar and its derivative do not diverge for any solutions implying that there is no real singularity at the origin of the gravitating string.

The solutions we obtain when varying the parameter $W'(0)$ in addition to the other parameters mentioned above have the same $R'(0)$ (same parameter space as shown in figure 4). The difference lies in the value of $P''(0)$, which is simply the value of $P''(0, W'(0) = 1)$ multiplied by $W'(0)$. The solution surface for $P''(0)$ therefore is the same as figure 5 scaled by $W'(0)$. This completes the solution space. We can now compare solutions for which the only parameter varied is $W'(0)$. We find that they exhibit the same physical bending and therefore are equivalent (see figure 8 and 9). The angular deficit α_D, the ϕ component of the energy momentum tensor and the ϕ components of the Weyl tensor are observed to increase with $W'(0)$. This increase is by a factor α_D^2, which is consistent with a simple coordinate transformation. This demonstrates that the physics of the string spacetime does not depend on the choice of $W'(0)$ and therefore that the regularity at the axis is not a necessary condition.

6 Conclusion

We have shown that the Einstein-Euler-Lagrange equations describing a gravitating cosmic string can be solved simultaneously and accurately using a Taylor series method. This method allows us to also study the physical properties of the string solutions for any set of initial parameters we wish to examine. The question of regularity at the axis of the string was resolved in this work using physical arguments. The principal outcome of the work described in this paper is that there does occur significant gravitational lensing for many

of the solutions found. These strings may have important consequences in cosmology since they now are gravitating and are true gravitational lenses.

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Figure 1: The magnitude of the scalar and gauge field as a function of r and the difference between an "acceptable" and "unacceptable" solutions.

Figure 2: Contour plot of the measured deviation from the correct asymptotic structure around the "acceptable" solution with parameters $\alpha = 1.0$, $\eta = 0.19947106$, $R'(0) = 1.4586085$ and $P''(0) = -4$. The contours are plotted for values of the deviation of 0, 0.5, 1, 2, 5, 10, 20 and 20000. There is a deep valley very near the "acceptable" solution were the deviations get smaller than in the surrounding regions.

Figure 3: Circular scans of the measured deviation shown in figure 2. Scans are drawn for circles of different radii around the "acceptable" solution. A complete circle is shown by an angle going from 0 to 2π as shown on the abscissa. The narrow valley of deviations is more apparent in this figure.

Figure 4: Surface representing the subset of "acceptable" solution. The surface covers a range in α of 0.001 to 2.5 and a range in η of 0.0 to 0.2. This plot shows the value of $R'(0)$ for a specific choice of α and η parameters that will give an "acceptable" solution.

Figure 5: For the same range as given in figure 4, this plot gives the value of $P''(0)$ for a specific choice of α and η parameters that will give an "acceptable" solution.

Figure 6: Shown here are different "acceptable" solutions and some of their physical characteristics. The first row shows the solution $\alpha = 1.0$, $\eta = 0.19947106$, $W'(0) = 1.0$, $R'(0) = 1.4586085$ and $P''(0) = -4$, the second row $\alpha = 1.0$, $\eta = 0.2820947$, $W'(0) = 1.0$, $R'(0) = 1.213061694$ and $P''(0) = -4.0$, the third row $\alpha = 0.319$, $\eta = 0.15$, $W'(0) = 1.0$, $R'(0) = 1.081678$ and $P''(0) = -2.848065$ and the last row $\alpha =1.5$, $\eta = 0.01$, $W'(0) = 1.0$, $R'(0) = 1.9746463$ and $P''(0) = -4.5213392$.

Figure 7: The orbit of a photon in the equatorial plane of a cosmic string, plotted in terms of the cylindrical coordinates r and ϕ, for selected values of the closest approach radius of the orbit. The orbits depicted here are for the solution with parameters $\alpha = 2.0$, $\eta = 0.15$, $W'(0) = 1.0$, $R'(0) = 1.9955592$ and $P''(0) = -4.923572$.

Figure 8: The orbit of a photon are shown here for the solution with $\alpha = 1.0$, $\eta = 0.19947106$, $W'(0) = 1.0$, $R'(0) = 1.4586085$ and $P''(0) = -4.0$.

Figure 9: The orbit of a photon are shown for the solution with $W'(0) \neq 1.0$, that is with $\alpha = 1.0$, $\eta = 0.19947106$, $W'(0) = 0.5$, $R'(0) = 1.4586085$ and $P''(0) = -2.0$. There is no difference between this figure and the previous one with $W'(0) = 1.0$.